MOLLIFICATION OF $\mathcal{D}$-SOLUTIONS TO FULLY NONLINEAR PDE SYSTEMS

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Abstract. In a recent paper the author has introduced a new theory of generalised solutions which applies to fully nonlinear PDE systems of any order and allows the interpretation of merely measurable maps as solutions. This approach is duality-free and builds on the probabilistic representation of limits of difference quotients via Young measures over certain compactifications of the "state space". Herein we establish a systematic regularisation scheme of this notion of solution which, by analogy, is the counterpart of the usual mollification by convolution of weak solutions and of the mollification by sup/inf convolutions of viscosity solutions.

1. Introduction

Let $p,n,N,M \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ an open set and

$$F \ : \ \Omega \times \Big(\mathbb{R}^N \times \mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nn^p}_s\Big) \longrightarrow \mathbb{R}^M \tag{1.1}$$

a Carathéodory mapping. Here $\mathbb{R}^{Nn}$ denotes the space of $N\times n$ matrices wherein the gradient matrix

$$Du(x) \ = \ \big(D_i u_\alpha(x)\big)^{\alpha=1,\ldots,N}_{i=1,\ldots,n}$$

of (smooth) maps $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ is valued, whilst $\mathbb{R}^{Nn^p}_s$ denotes the space of symmetric tensors

$$\Big\{\mathbf{X} \in \mathbb{R}^{Nn^p} \ \big| \ \mathbf{X}_{\alpha i_1\ldots i_a \ldots i_b \ldots i_p} = \mathbf{X}_{\alpha i_1\ldots i_b \ldots i_a \ldots i_p} \ ,$$
$$\alpha = 1,...,N, \ i_1,...,i_p \in \{1,...,n\}, \ 1 \leq a \leq b \leq p\Big\}$$

wherein the pth order derivative

$$D^p u(x) \ = \ \Big(D^p_{i_1\ldots i_p} u_\alpha(x)\Big)^{\alpha=1,\ldots,N}_{i_1,\ldots,i_p \in \{1,\ldots,n\}}$$

is valued. Obviously, $D_i \equiv \partial/\partial x_i$, $x = (x_1,...,x_n)^\top$, $u = (u_1,...,u_N)^\top$ and $\mathbb{R}^{Nn^1}_s = \mathbb{R}^{Nn}$. In the recent paper [K8] we introduced a new theory of generalised solutions which allows for merely measurable maps to be rigorously interpreted and studied as solutions of systems with even discontinuous coefficients and without requiring any structural assumptions (like ellipticity or hyperbolicity). Namely, our approach applies to measurable solutions $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ of the pth order system

$$F\Big(x, u(x), D^{[p]}u(x)\Big) \ = \ 0, \quad x \in \Omega, \tag{1.2}$$

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where

$$D^{[p]}u \,:=\, \big(Du, D^2u, ..., D^pu\big)$$

denotes the *pth order Jet of* u. Since we do not assume that solutions must be locally integrable on Ω, the derivatives $Du, ..., D^pu$ may not have a classical meaning, not even in the distributional sense. Using this new approach, in the very recent papers [K8, K9, K10] we studied efficiently certain interesting problems arising in PDE theory and in vectorial Calculus of Variations which we discuss briefly at the end of the introduction.

In the present paper we are concerned with the development of a systematic method of mollification of generalised solutions to the fully nonlinear system (1.2) by constructing approximate smooth solutions to approximate systems. The mollification method we establish herein is the counterpart of the standard mollification by convolution which is the standard analytical tool in the study of weak solutions and to the so called sup/inf convolutions used in the theory of viscosity solutions of Crandall-Ishii-Lions (for a pedagogical introduction we refer to [K7]).

Our starting point for the definition of solution is not based either on standard duality considerations via integration-by-parts (the cornerstone of weak solutions) or on the maximum principle (the mechanism of the more recent method of viscosity solutions). Instead, we build on the probabilistic representation of limits of difference quotients by utilising *Young* measures, also known as parameterised measures. These are well-developed objects of abstract measure theory of great importance in Calculus of Variations and PDE theory (see e.g. [K8] and [E, P, FL, CFV, FG, V, KR]). In the present setting, a version of Young measures is utilised in order to define generalised solutions of (1.2) by applying them to the difference quotients of the candidate solution. The essential idea restricted to the first order case $p=1$ of (1.2) goes as follows: let $u \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^N)$ be a strong solution to

$$F\big(x, u(x), Du(x)\big) \,=\, 0, \qquad \text{a.e. } x \in \Omega. \tag{1.3}$$

Let us now rewrite (1.3) in the following unconventional fashion:

$$\sup_{X \in \, \text{supp}(\delta_{Du(x)})} \big|F\big(x, u(x), X\big)\big| \,=\, 0, \qquad \text{a.e. } x \in \Omega.$$

That is, we understand the gradient $Du : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{Nn}$ as a probability-valued map given by the Dirac measure at the gradient

$$\delta_{Du} \;:\; \Omega \subseteq \mathbb{R}^n \longrightarrow \mathscr{P}(\mathbb{R}^{Nn}), \quad x \longmapsto \delta_{Du(x)},$$

in hopes of relaxing the requirement to have a concentration measure. The goal is to allow instead general probability-valued maps arising as limits of the difference quotients for nonsmooth maps. Indeed, if $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ is only *measurable*, we consider the probability-valued mappings

$$\delta_{D^{1,h}u} \;:\; \Omega \subseteq \mathbb{R}^n \longrightarrow \mathscr{P}\big(\overline{\mathbb{R}}^{Nn}\big), \quad x \longmapsto \delta_{D^{1,h}u(x)},$$

where $D^{1,h}$ is the usual difference quotients operator and $\overline{\mathbb{R}}^{Nn}$ is the Alexandroff 1-point compactification $\mathbb{R}^{Nn} \cup \{\infty\}$. Namely, we view $D^{1,h}u$ as an element of the space of Young measures $\mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn}\big)$ (see the next Section 2 for the precise definitions). By using that $\mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn}\big)$ is weakly* compact, there always exist

probability-valued mappings $\mathcal{D}u \in \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn}\big)$ such that along infinitesimal subsequences $(h_\nu)_1^\infty$ we have

$$\delta_{D^{1,h_\nu}u} \overset{*}{\rightharpoonup} \mathcal{D}u \quad \text{in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn}\big), \qquad \text{as } \nu \to \infty \tag{1.4}$$

(even if u is merely measurable). Then, we require[1]

$$\sup_{X \in \, \mathrm{supp}_*(\mathcal{D}u(x))} \big|F\big(x, u(x), X\big)\big| \,=\, 0, \qquad \text{a.e. } x \in \Omega, \tag{1.5}$$

for any "diffuse" gradient $\mathcal{D}u$, where

$$\mathrm{supp}_*(\mathcal{D}u(x)) \,:=\, \mathrm{supp}(\mathcal{D}u(x)) \setminus \{\infty\}.$$

Since (1.4) and (1.5) are *independent* of the regularity of u, they can be taken as a notion of **diffuse derivatives** of the measurable map $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ and of **$\mathcal{D}$-solutions** to the PDE system (1.3) respectively. If u happens to be weakly differentiable, then we have $\mathcal{D}u = \delta_{Du}$ a.e. on Ω and we reduce to strong solutions. Except for a small further technical generalisation (we may need to take special difference quotients depending on F), (1.4) and (1.5) comprise our notion of generalised solutions in the first order case of (1.2).

This paper is organised as follows. The introduction is followed by Section 2 which is a quick review of the main points of [K8] necessary for this work. The main results of this paper are in Section 3 (Theorem 11 and Corollary 12) and establish the main properties of our approximations. The technical core of our analysis is contained in the preparatory Lemma 10. We expect the analytical results established herein to play a prominent role in future developments of the theory, but we refrain from providing any immediate applications in this paper.

We conclude this introduction with some results recently obtained by using the technology of $\mathcal{D}$-solutions. Our motivation to introduce them primarily comes from the necessity to study the recently discovered equations arising in vectorial Calculus of Variations in the space L^∞, that is for variational problem related to functionals

$$E_\infty(u, \Omega) \,:=\, \big\|H(\cdot, u, Du)\big\|_{L^\infty(\Omega)} \tag{1.6}$$

applied to Lipschitz maps $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ (for an introduction to the topic we refer to [C, BEJ, K7]). In the simplest case of $H(\cdot, u, Du) = |Du|^2$, the analogue of the Euler-Lagrange equation is the ∞-Laplace system:

$$\Delta_\infty u \,:=\, \Big(Du \otimes Du + |Du|^2 [Du]^\perp \otimes I\Big) : D^2 u \,=\, 0, \tag{1.7}$$

where $[Du]^\perp := \mathrm{Proj}_{(R(Du))^\perp}$. In index form (1.7) reads

$$\sum_{\beta=1}^{N} \sum_{i,j=1}^{n} \Big(D_i u_\alpha \, D_j u_\beta + \, |Du|^2 [Du]^\perp_{\alpha\beta} \, \delta_{ij}\Big) \, D^2_{ij} u_\beta \,=\, 0, \qquad \alpha = 1, ..., N$$

and $[Du]^\perp$ is the orthogonal projection on the orthogonal complement of the range of Du. The vectorial case of the theme has been pioneered by the author in a series of recent papers [K1]-[K6], while the scalar case is relatively standard by now and has been pioneered by Aronsson in the 1960s ([K7]). In the paper [K8] we studied the Dirichlet problem for (1.7), while in [K9] we studied the Dirichlet problem for the system arising from the general functional (1.6) for $n = 1$. In [K9]

[1]The version of the definition we are using herein is different from the one we put foremost in [K8]-[K10] because this simplifies the proofs that follow. In [K8] we proved several equivalent formulations of the same notion which we will not utilise, so we take this version as primary here.

we also considered the Dirichlet problem for fully nonlinear 2nd order degenerate elliptic systems and in [K10] we considered the problem of equivalence between distributional and $\mathcal{D}$-solutions for linear symmetric hyperbolic systems.

2. A quick guide to $\mathcal{D}$-solutions for fully nonlinear systems

2.1. **Preliminaries.** We begin with some basic material needed in the rest of the paper.

Basics. The constants $n, N \in \mathbb{N}$ will always denote the dimensions of the domain and the target of our candidate solutions $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ defined over an open set. Such mappings will always be understood as being extended by zero on $\mathbb{R}^n \setminus \Omega$. Unless indicated otherwise, Greek indices $\alpha, \beta, \gamma, ...$ will run in $\{1, ..., N\}$ and latin indices $i, j, k, ...$ (perhaps indexed $i_1, i_2, ...$) will run in $\{1, ..., n\}$, even when their range of summation may not be given explicitly. The norm symbols $|\cdot|$ will always mean the Euclidean ones, whilst Euclidean inner products will be denoted by either "$\cdot$" on $\mathbb{R}^n, \mathbb{R}^N$ or by "$:$" on tensor spaces. For example, on $\mathbb{R}^{Nn^p}_s$ we have

$$|\mathbf{X}|^2 \,=\, \sum_{\alpha, i_1, ... i_p} \mathbf{X}_{\alpha i_1 ... i_p} \mathbf{X}_{\alpha i_1 ... i_p} \,\equiv\, \mathbf{X} : \mathbf{X}.$$

Our measure theoretic and function space notation is either standard as e.g. in [E, E2] or self-explanatory. For example, the modifier "measurable" will always mean "Lebesgue measurable", the Lebesgue measure will be denoted by $|\cdot|$, the characteristic function of a set E by χ_E, the L^p spaces of maps $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ by $L^p(\Omega, \mathbb{R}^N)$, etc.

We will systematically use the Alexandroff 1-point compactification of $\mathbb{R}^{Nn^p}_s$. The metric topology on it will be the standard one which makes it isometric to the sphere of equal dimension (via the stereographic projection which identifies the the north pole with infinity $\{\infty\}$). It will denoted by

$$\overline{\mathbb{R}}^{Nn^p}_s \,:=\, \mathbb{R}^{Nn^p}_s \cup \{\infty\}.$$

We also note that balls taken in $\mathbb{R}^{Nn^p}_s$ (which we will view as a metric vector space isometrically contained into $\overline{\mathbb{R}}^{Nn^p}_s$) will be understood as the Euclidean.

Young Measures. Let E be a measurable subset of $\mathbb{R}^n$ and $\mathbb{K}$ a compact subset of some Euclidean space, which we will later take to be $\overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s$.

Definition 1 (Young Measures)**.** The set of Young Measures $\mathscr{Y}(E, \mathbb{K})$ consists of the probability-valued mappings

$$\vartheta \;:\; E \longrightarrow \mathscr{P}(\mathbb{K}), \quad x \longmapsto \vartheta(x),$$

which are measurable in the following weak* (i.e. pointwise) sense: for any continuous function $\Psi \in C^0(\mathbb{K})$, the function $E \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$\int_{\mathbb{K}} \Psi(X)\, d[\vartheta(\cdot)](X) \;\;:\;\;\; x \longmapsto \int_{\mathbb{K}} \Psi(X)\, d[\vartheta(x)](X)$$

is (Lebesgue) measurable.

The set $\mathscr{Y}(E, \mathbb{K})$ can be identified with a subset of the unit sphere of a certain L^∞ space and this provides very useful compactness and other properties. Consider the L^1 space of Bochner integrable maps

$$L^1\big(E, C^0(\mathbb{K})\big)$$

which are valued in the separable space $C^0(\mathbb{K})$ of continuous functions over $\mathbb{K}$. For background material on these spaces we refer e.g. to [FL, Ed, F, V]. The elements of $L^1\big(E, C^0(\mathbb{K})\big)$ coincide with the Carathéodory functions

$$\Phi \ : \ E \times \mathbb{K} \longrightarrow \mathbb{R}, \quad (x, X) \mapsto \Phi(x, X)$$

which satisfy

$$\|\Phi\|_{L^1(E,C^0(\mathbb{K}))} := \int_E \big\|\Phi(x,\cdot)\big\|_{C^0(\mathbb{K})}\, dx \, < \, \infty$$

in the sense that each such Φ induces a map $E \ni x \mapsto \Phi(x,\cdot) \in C^0(\mathbb{K})$. By Carathéodory functions we mean that for every $X \in \mathbb{K}$ the function $x \mapsto \Phi(x, X)$ is measurable and for a.e. $x \in E$ the function $X \mapsto \Phi(x, X)$ is continuous. The Banach space $L^1\big(E, C^0(\mathbb{K})\big)$ is separable and by using the duality $\big(C^0(\mathbb{K})\big)^* = \mathcal{M}(\mathbb{K})$, it can be shown that (see e.g. [FL])

$$\big(L^1\big(E, C^0(\mathbb{K})\big)\big)^* \, = \, L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big).$$

The dual space $L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big)$ consists of measure-valued maps

$$E \ \ni \ x \longmapsto \vartheta(x) \ \in \mathcal{M}(\mathbb{K})$$

which are weakly* measurable and the norm of the space is given by

$$\|\vartheta\|_{L^\infty_{w^*}(E,\mathcal{M}(\mathbb{K}))} := \underset{x\in E}{\operatorname{ess\,sup}}\, \|\vartheta(x)\|\,(\mathbb{K}).$$

Here "$\|\cdot\|(\mathbb{K})$" denotes the total variation on $\mathbb{K}$. The duality pairing between the spaces

$$\langle \cdot,\cdot\rangle \ : \ \ L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big) \times L^1\big(E, C^0(\mathbb{K})\big) \longrightarrow \mathbb{R}$$

is given by

$$\langle \vartheta, \Phi\rangle := \int_E \int_{\mathbb{K}} \Phi(x, X)\, d[\vartheta(x)](X)\, dx.$$

Then, the set of Young measures can be identified with a subset of the unit sphere of $L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big)$:

$$\mathscr{Y}(E,\mathbb{K}) \, = \, \Big\{\vartheta \, \in \, L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big) \, : \, \vartheta(x) \in \mathscr{P}(\mathbb{K}), \text{ for a.e. } x \in E\Big\}.$$

Since $L^1\big(E, C^0(\mathbb{K})\big)$ is separable, the unit ball of $L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big)$ is sequentially weakly* compact. Hence, for any bounded sequence $(\vartheta^m)_1^\infty \subseteq L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big)$, there is a limit map ϑ and a subsequence of m's along which $\vartheta^m \overset{*}{\rightharpoonup} \vartheta$ as $m \to \infty$.

Remark 2 (Properties of $\mathscr{Y}(E,\mathbb{K})$)**.** The following facts about Young measures will be extensively used hereafter (the proofs can be found e.g. in [FG]):

i) [**Weak* compactness of Y.M.**] The set of Young measures is convex and sequentially compact in the weak* topology induced from $L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big)$.

ii) [**Mappings as Y.M.**] The set of measurable maps $v : E \subseteq \mathbb{R}^n \longrightarrow \mathbb{K}$ can be imbedded into $\mathscr{Y}(E,\mathbb{K})$ via the mapping $v \mapsto \delta_v$ given by $\delta_v(x) := \delta_{v(x)}$.

The next lemma is a minor variant of a classical result (see [K8, FG, FL]) but it plays a fundamental role in our setting because it guarantees the compatibility of strong solutions with $\mathcal{D}$-solutions.

Lemma 3. *Suppose $E \subseteq \mathbb{R}^n$ is a measurable set and $U^\nu, U^\infty : E \longrightarrow \mathbb{K}$ are measurable maps, $\nu \in \mathbb{N}$. Then, there exist subsequences $(\nu_k)_1^\infty$, $(\nu_l)_1^\infty$:*

$$\text{(1)} \quad \delta_{U^\nu} \overset{*}{\rightharpoonup} \delta_{U^\infty} \ \text{in } \mathscr{Y}(E,\mathbb{K}) \quad \Longrightarrow \quad U^{\nu_k} \longrightarrow U^\infty \quad \text{a.e. on } E.$$

$$\text{(2)} \quad U^\nu \longrightarrow U^\infty \quad \text{a.e. on } E \quad \Longrightarrow \quad \delta_{U^{\nu_l}} \overset{*}{\rightharpoonup} \delta_{U^\infty} \ \text{in } \mathscr{Y}(E,\mathbb{K}).$$

General frames, derivative expansions, difference quotients. In what follows we will consider non-standard orthonormal frames of $\mathbb{R}^{Nn^p}_s$ and write derivatives $D^p u$ with respect to them. This generalisation is irrelevant to the mollification results we establish herein but it was absolutely essential for the existence-uniqueness results we established in [K8]-[K10]. In any case, these bases will not appear explicitly anywhere in the proofs and they will not imply any technical ramifications.

Let $\{E^1,...,E^N\}$ be an orthonormal frame of $\mathbb{R}^N$ and suppose that for each $\alpha = 1,...,N$ we have an orthonormal frame $\{E^{(\alpha)1},...,E^{(\alpha)n}\}$ of $\mathbb{R}^n$. Given such bases, we will equip the space $\mathbb{R}^{Nn^p}_s$ with the following induced orthonormal base:

$$\mathbb{R}^{Nn^p}_s = \text{span}\big[\big\{E^{\alpha i_1 \dots i_p}\big\}\big], \qquad E^{\alpha i_1 \dots i_p} := E^\alpha \otimes \Big(E^{(\alpha)i_1} \vee ... \vee E^{(\alpha)i_p}\Big) \tag{2.1}$$

where

$$a \vee b := \frac{1}{2}\Big(a \otimes b + b \otimes a\Big), \quad a, b \in \mathbb{R}^n, \tag{2.2}$$

is the symmetrised tensor product. Given such frames, let

$$D^p_{E^{(\alpha)i_p} \dots E^{(\alpha)i_1}} = D_{E^{(\alpha)i_p}} \cdots D_{E^{(\alpha)i_1}}$$

denote the usual pth order directional derivative along the respective directions. Then, the pth order derivative $D^p u$ of a map $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ can be expressed as

$$\begin{aligned} D^p u \ &= \sum_{\alpha, i_1, \dots, i_p} \Big(E^{\alpha i_1 \dots i_p} : D^p u\Big)\, E^{\alpha i_1 \dots i_p} = \\ &= \sum_{\alpha, i_1, \dots, i_p} \Big(D^p_{E^{(\alpha)i_1} \dots E^{(\alpha)i_p}}(E^\alpha \cdot u)\Big)\, E^{\alpha i_1 \dots i_p}. \end{aligned} \tag{2.3}$$

We will use the following compact notation for the (formal) Taylor expansion around a point $x \in \Omega$:

$$u(z) = \sum_{p=1}^\infty \frac{1}{p!}\, D^p u(x) : (z-x)^{\otimes p}.$$

The notation "$\otimes p$" stands for the pth tensor power and ":" is the obvious contraction of indices which in index form reads

$$u_\alpha(z) = \sum_{p=1}^\infty \frac{1}{p!} \sum_{i_1, \dots i_p = 1}^n D^p_{i_1 \dots i_p} u_\alpha(x)\, (z-x)_{i_1} \dots (z-x)_{i_p}.$$

Expansions analogous to (2.3) will also be applied to difference quotients which play a crucial role in our approach. Given $a \in \mathbb{R}^n$ with $|a| = 1$ and $h \in \mathbb{R} \setminus \{0\}$, the 1st order difference quotient of u along the direction a at x will be denoted by

$$D^{1,h}_a u(x) := \frac{u(x+ha) - u(x)}{h}. \tag{2.4}$$

By iteration, if $h_1, ..., h_p \neq 0$ the pth order difference quotient along $a_1, ..., a_p$ is

$$D^{p,h_p \dots h_1}_{a_p \dots a_1} u \,:=\, D^{1,h_p}_{a_p}\Big(\cdots \big(D^{1,h_1}_{a_1} u\big)\Big). \tag{2.5}$$

We now introduce difference quotients taken with respect to frames as in (2.1).

Definition 4 (Difference quotients)**.** Let $\{E^1, ..., E^N\}$ be an orthonormal frame of $\mathbb{R}^N$ and let also $\{E^{(\alpha)1}, ..., E^{(\alpha)n}\}$ be for each $\alpha = 1, ..., N$ an orthonormal frame of $\mathbb{R}^n$, while for any $p \in \mathbb{N}$ the tensor space $\mathbb{R}^{Nn^p}_s$ is equipped with the frame (2.1). Given any **vector-indexed** infinitesimal sequence

$$(h_{\overline{m}})_{\overline{m}\in\mathbb{N}^p} \subseteq \big(\mathbb{R}\setminus\{0\}\big)^p, \quad \overline{m} = (m^1, ..., m^p), \quad h_{m^q} \to 0 \ \text{ as } m^q \to \infty,$$

we define the *pth order difference quotients* of the measurable mapping $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ (with respect to the fixed reference frames) arising from $(h_{\overline{m}})_{\overline{m}\in\mathbb{N}^p}$ as the family of maps

$$D^{p,h_{\overline{m}}} u \;:\; \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{Nn^p}_s, \qquad \overline{m} \in \mathbb{N}^p,$$

each of which is given by

$$D^{p,h_{\overline{m}}} u \,:=\, \sum_{\alpha, i_1, ..., i_p} \Big[D^{p,h_{m^p}\dots h_{m^1}}_{E^{(\alpha)i_p}\dots E^{(\alpha)i_1}} (E^\alpha \cdot u)\Big]\, E^{\alpha i_1 \dots i_p}.$$

The notation in the bracket above is as in (2.4), (2.5). Further, given any **matrix-indexed** infinitesimal sequence

$$(h_{\underline{m}})_{\underline{m}\in\mathbb{N}^{p\times p}} \subseteq \big(\mathbb{R}\setminus\{0\}\big)^{p\times p}, \quad \underline{m} = \begin{bmatrix} m^1_1 & 0 & 0 \; ... \; 0 \\ m^1_2 & m^2_2 & 0 \; ... \; 0 \\ \vdots & & \ddots \; \vdots \\ m^1_p & m^2_p & ... \quad m^p_p \end{bmatrix}, \ \ h_{m^q_p} \to 0 \text{ as } m^q_p \to \infty,$$

we will denote its nonzero row elements by

$$\underline{m}_q \,:=\, (m^1_q, ..., m^q_q) \ \in \mathbb{N}^q, \quad q \,=\, 1, ..., p,$$

and we define the **pth order Jet $D^{[p],h_{\underline{m}}} u$ of difference quotients of u** (with respect to the reference frames) arising from $(h_{\underline{m}})_{\underline{m}\in\mathbb{N}^{p\times p}}$ as the family of maps

$$D^{[p],h_{\underline{m}}} u \;:\; \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{Nn} \times ... \times \mathbb{R}^{Nn^p}_s, \qquad \underline{m} \in \mathbb{N}^{p\times p},$$

each of which is given by

$$D^{[p],h_{\underline{m}}} u \,:=\, \Big(D^{1,h_{\underline{m}_1}} u, \dots, D^{p,h_{\underline{m}_p}} u\Big).$$

Definition 5 (Multi-indexed convergence)**.** If $\underline{m} \in \mathbb{N}^{p\times p}$ is a lower trigonal matrix of indices as above, the expression "$\underline{m} \to \infty$" will by definition mean successive convergence with respect to each index separately in the following order:

$$\lim_{\underline{m}\to\infty} \;:=\; \lim_{m^p_p\to\infty} \dots \lim_{m^2_2\to\infty} \lim_{m^1_2\to\infty} \lim_{m^1_1\to\infty}.$$

2.2. **Main definitions and some analytic properties.**

Definition 6 (Diffuse Jets)**.** Suppose we have fixed some reference frames as in Definition 4. For any measurable map $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$, we define the **diffuse** p**th order Jets** $\mathcal{D}^{[p]}u$ **of** u as the following subsequential weak* limits:

$$\delta_{D^{[p],h_{\underline{m}}}u} \overset{*}{\rightharpoonup} \mathcal{D}^{[p]}u \quad \text{ in } \mathscr{Y}\Big(\Omega, \overline{\mathbb{R}}^{Nn} \times ... \times \overline{\mathbb{R}}^{Nn^p}_s\Big),$$

which arise as $\underline{m} \to \infty$ along multi-indexed infinitesimal subsequences.

As a consequence of the separate convergence, the pth order Jet is always a (fibre) product Young measure: $\mathcal{D}^{[p]}u = \mathcal{D}u \times \cdots \times \mathcal{D}^p u$.

Next is the central notion of generalised solution. We will use the notation $\underline{\mathbf{X}} \equiv (\mathbf{X}_1, ..., \mathbf{X}_p)$ for points in $\overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s$ and also the symbol "supp_*" to denote the *reduced support* of a probability measure $\vartheta \in \mathscr{P}\big(\overline{\mathbb{R}}^{Nn} \times ... \times \overline{\mathbb{R}}^{Nn^p}_s\big)$ off "infinity":

$$\text{supp}_*(\vartheta) \,:=\, \text{supp}(\vartheta) \cap \Big(\mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nn^p}_s\Big).$$

Definition 7 ($\mathcal{D}$-solutions for pth order systems)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set and

$$F \;:\; \Omega \times \Big(\mathbb{R}^N \times \mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nn^p}_s\Big) \longrightarrow \mathbb{R}^M$$

a Carathéodory mapping. Assume also that we have fixed some reference frames as in Definition 4 and consider the pth order PDE system

$$F\Big(x, u(x), D^{[p]}u(x)\Big) \,=\, 0, \quad x \in \Omega. \tag{2.6}$$

We say that the measurable map $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ is a $\mathcal{D}$**-solution of** (2.6) when for any diffuse pth order Jet $\mathcal{D}^{[p]}u$ of u arising from any infinitesimal multi-indexed sequence $(h_{\underline{m}})_{\underline{m} \in \mathbb{N}^{p \times p}}$ (Definition 6) we have

$$\sup_{\underline{\mathbf{X}} \in \, \text{supp}_*(\mathcal{D}^{[p]}u(x))} \Big|F\big(x, u(x), \underline{\mathbf{X}}\big)\Big| \,=\, 0, \qquad \text{a.e. } x \in \Omega.$$

We now consider the consistency of the $\mathcal{D}$-notions with the strong/classical notions of solution. For more details we refer to [K8] and also to [K9, K10]. In general, diffuse derivatives may be nonunique for nonsmooth maps. However, as the next simple consequence of Lemma 3 shows, they are compatible with weak derivatives and a fortiori with classical derivatives:

Lemma 8 (Compatibility of weak and diffuse derivatives)**.** *If $u \in W^{p,1}_{loc}(\Omega, \mathbb{R}^N)$, then the pth order diffuse Jet $\mathcal{D}^{[p]}u$ is unique and for any $k \in \mathbb{N}$ we have*

$$\mathcal{D}^{[p+k]}u \,=\, \delta_{(Du,...,D^pu)} \times \mathcal{D}^{p+1} \times ... \times \mathcal{D}^{p+k}u, \quad \textit{a.e. on } \Omega.$$

The next result asserts the plausible fact that $\mathcal{D}$-solutions are compatible with strong solutions. Its proof is an immediate consequence of Lemma 8.

Proposition 9 (Compatibility of strong and $\mathcal{D}$-solutions)**.** *Let F be a Carathéodory map as in* (1.1) *and $u \in W^{p,1}_{loc}(\Omega, \mathbb{R}^N)$. Consider the pth order PDE system*

$$F\Big(x, u(x), D^{[p]}u(x)\Big) \,=\, 0, \quad x \in \Omega.$$

Then, u is a $\mathcal{D}$-solution on Ω if and only if u is a strong a.e. solution on Ω.

Lemma 8 and Proposition 9 remain true if u is merely p-times differentiable in measure, a notion weaker than approximate differentiability (see [K8, AM]). For more details on the material of this section (e.g. analytic properties, equivalent formulations of Definition 7, etc) we refer to [K8]-[K10].

3. Mollification of $\mathcal{D}$-solutions to fully nonlinear systems

We begin with the next result which is the main technical core of our constructions. Our method of proof is inspired by the paper of Alberti [A].

Lemma 10 (Construction of the approximations). *Let $\Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be a measurable map and $p \in \mathbb{N}$. Then, for any $\varepsilon > 0$ and any multi-index $\underline{m} \in \mathbb{N}^{p\times p}$ as in Definition 4 , there exist a measurable set $E_{\varepsilon,\underline{m}} \subseteq \Omega$ and a smooth map $u^{\varepsilon,\underline{m}} \in C^\infty_0(\Omega,\mathbb{R}^N)$ such that*

$$
\left\{
\begin{array}{l}
\big|E_{\varepsilon,\underline{m}}\big| \,\leq\, \varepsilon,\\
\big\|u \,-\, u^{\varepsilon,\underline{m}}\big\|_{L^\infty(\Omega\setminus E_{\varepsilon,\underline{m}})} \,\leq\, \varepsilon,\\
\Big\|D^{[p],h_{\underline{m}}}u \,-\, D^{[p]}u^{\varepsilon,\underline{m}}\Big\|_{L^\infty(\Omega\setminus E_{\varepsilon,\underline{m}})} \,\leq\, \varepsilon,
\end{array}
\right.
\tag{3.1}
$$

where $D^{[p],h_{\underline{m}}}u$ is the pth order Jet of difference quotients of u. If moreover $|\Omega| < \infty$, then $u^{\varepsilon,\underline{m}} \in C^\infty_c(\Omega,\mathbb{R}^N)$. Finally, if $u \in L^r(\Omega,\mathbb{R}^N)$ for some $r \in [1,\infty)$, then we also have

$$
\big\|u - u^{\varepsilon,\underline{m}}\big\|_{L^r(\Omega)} \,\leq\, \varepsilon.
\tag{3.2}
$$

The reader can easily be convinced that even if $u \in L^1_{\text{loc}}(\Omega,\mathbb{R}^N)$, the standard mollifier $u * \eta^\varepsilon$ of u does not satisfy these approximation properties (the best we can get is approximation in dual spaces, not almost uniform on Ω).

Proof of Lemma 10. Let $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be a given measurable map (extended on $\mathbb{R}^n \setminus \Omega$ by zero). Let $D^{[p],h_{\underline{m}}}u$ be the Jet of pth order difference quotients of u where the multi-index $\underline{m} \in \mathbb{N}^{p\times p}$ is fixed. We also fix $\varepsilon > 0$.

Step 1. We may assume that Ω has finite measure. This hypothesis does not harm generality for the following reason: assuming we have established (3.1), (3.2) on subdomains of Ω which have finite measure, we can fill Ω a.e. by disjoint open cubes $(\Omega_i)_1^\infty$ such that $\big|\Omega \setminus \big(\cup_1^\infty \Omega_i\big)\big| = 0$ and on each Ω_i (3.1) holds with $2^{-i}\varepsilon$ instead of ε for respective sequences of functions $(u^{\varepsilon,\underline{m},i})_{i=1}^\infty$ and sets $(E_{\varepsilon,\underline{m},i})_{i=1}^\infty$. Then, we define $u^{\varepsilon,\underline{m}} \in C^\infty_0(\Omega,\mathbb{R}^N)$ and $E_{\varepsilon,\underline{m}} \subseteq \Omega$ with $|E_{\varepsilon,\underline{m}}| \leq \varepsilon$ by taking

$$
u^{\varepsilon,\underline{m}}|_{\Omega_i} \,:=\, u^{\varepsilon,\underline{m},i}, \qquad E_{\varepsilon,\underline{m}} \,:=\, \cup_{i=1}^\infty E_{\varepsilon,\underline{m},i}.
$$

The conclusion of Lemma 10 then follows.

Step 2. We now show there exists a measurable set and smooth maps

$$
F_{\varepsilon,\underline{m}} \,\subseteq\, \Omega, \qquad \big\{U^{q,\varepsilon,\underline{m}}\big\}_{q=0}^p, \quad U^{q,\varepsilon,\underline{m}} \,\in C^\infty_c\big(\Omega,\mathbb{R}^{Nn^q}_s\big),
$$

such that

$$
\left\{
\begin{array}{l}
\big|F_{\varepsilon,\underline{m}}\big| \,\leq\, \varepsilon,\\
\big\|u \,-\, U^{0,\varepsilon,\underline{m}}\big\|_{L^\infty(\Omega\setminus F_{\varepsilon,\underline{m}})} \,\leq\, \varepsilon,\\
\big\|D^{q,h_{\underline{m}}}u \,-\, U^{q,\varepsilon,\underline{m}}\big\|_{L^\infty(\Omega\setminus F_{\varepsilon,\underline{m}})} \,\leq\, \varepsilon, \quad q = 1,...,p.
\end{array}
\right.
\tag{3.3}
$$

Indeed, let us define

$$V \;:=\; \big(u, D^{[p],h_{\underline{m}}}u\big) \;:\; \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N \times \mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nn^p}_s \tag{3.4}$$

and for any $R > 0$ we consider the truncation

$$T^R(\xi) \;:=\; \begin{cases} \xi, & \text{if } |\xi| < R, \\ R\dfrac{\xi}{|\xi|}, & \text{if } |\xi| \geq R. \end{cases} \tag{3.5}$$

Then, we have that $|T^R(V)| \in (L^1 \cap L^\infty)(\Omega)$. Since $T^R(V) \longrightarrow V$ a.e. on Ω as $R \to \infty$, we have that $T^R(V) \longrightarrow V$ in measure on Ω as $R \to \infty$. By the identity

$$\big|T^R(V) - V\big| \;=\; \big(|V| - R\big)\chi_{\{|V| \geq R\}},$$

for any $R > \varepsilon$ we have that

$$\begin{aligned} \big|\big\{T^R(V) \neq V\big\}\big| \;&=\; \big|\big\{|V| \geq R\big\}\big| \\ &\leq\; \Big|\big\{|V| \geq R/2\big\} \cap \big\{|V| > (R+\varepsilon)/2\big\}\Big| \\ &=\; \Big|\Big\{\big(|V| - R/2\big)\chi_{\{|V| \geq R/2\}} > \varepsilon/2\Big\}\Big| \\ &=\; \Big|\Big\{\big|T^{R/2}(V) - V\big| > \varepsilon/2\Big\}\Big| \\ &\longrightarrow\; 0, \end{aligned}$$

as $R \to \infty$. Hence, for $R(\varepsilon) > 0$ large enough, there is a measurable set $A_\varepsilon \subseteq \Omega$ such that

$$|A_\varepsilon| \leq \varepsilon/2, \qquad T^{R(\varepsilon)}(V) \;=\; V \text{ on } \Omega \setminus A_\varepsilon. \tag{3.6}$$

Further, we can find a sequence of smooth compactly supported maps $(V^{\varepsilon,k})_{k=1}^\infty$ such that for any $s \in [1,\infty)$,

$$\big|V^{\varepsilon,k} \,-\, T^{R(\varepsilon)}(V)\big| \longrightarrow 0, \quad \text{in } L^s(\Omega) \text{ and a.e. on } \Omega \text{ as } k \to \infty.$$

By Egoroff's theorem, the convergence is almost uniform on Ω as well. Hence, we can find a large enough $k(\varepsilon) \in \mathbb{N}$ and a measurable set $B_\varepsilon \subseteq \Omega$ such that

$$|B_\varepsilon| \leq \varepsilon/2, \qquad \big\|V^{\varepsilon,k(\varepsilon)} \,-\, T^{R(\varepsilon)}(V)\big\|_{L^\infty(\Omega \setminus B_\varepsilon)} \leq\; \varepsilon. \tag{3.7}$$

By (3.6) and (3.7), we conclude that by considering the measurable set $F_{\varepsilon,\underline{m}} \subseteq \Omega$ given by

$$F_{\varepsilon,\underline{m}} \;:=\; A_\varepsilon \cup B_\varepsilon$$

we have

$$\big|F_{\varepsilon,\underline{m}}\big| \leq \varepsilon, \qquad \big\|V^{\varepsilon,k(\varepsilon)} \,-\, V\big\|_{L^\infty(\Omega \setminus F_{\varepsilon,\underline{m}})} \leq\; \varepsilon.$$

Hence, (3.3) ensues because in view of (3.4) we may define

$$\left\{ \begin{aligned} U^{0,\varepsilon,\underline{m}} \;&:=\; \mathrm{Proj}_{\mathbb{R}^N}\big(V^{\varepsilon,k(\varepsilon)}\big), \\ U^{q,\varepsilon,\underline{m}} \;&:=\; \mathrm{Proj}_{\mathbb{R}^{Nn^q}_s}\big(V^{\varepsilon,k(\varepsilon)}\big), \qquad q = 1, ...p. \end{aligned} \right. \tag{3.8}$$

Step 3. We now establish (3.1) by using the previous step. Let $\omega_{\varepsilon,\underline{m}} \in C^0[0,\infty)$ be the joint modulus of continuity of the mappings $\{U^{q,\varepsilon,\underline{m}}\}_{q=0}^p$ of (3.3), i.e. $\omega_{\varepsilon,\underline{m}}(0) = 0$, $\omega_{\varepsilon,\underline{m}}$ is increasing and

$$\sum_{q=0,1,\dots,p} \Big|U^{q,\varepsilon,\underline{m}}(z) \,-\, U^{q,\varepsilon,\underline{m}}(y)\Big| \;\leq\; \omega_{\varepsilon,\underline{m}}\big(|z-y|\big), \qquad z, y \in \Omega. \tag{3.9}$$

We fix $\delta \in (0,1)$ that will be specified later and consider the grid $(\delta\mathbb{N})^n \subseteq \mathbb{R}^n$. Let also

$$\{Q_{\delta,i}\}_{i=1}^{\infty} := \Big\{\text{enumeration of open cubes whose vertices are on } (\delta\mathbb{N})^n\Big\}. \tag{3.10}$$

Fix also $\alpha \in (0,1)$ and consider

$$\{Q_{\alpha\delta,i}\}_{i=1}^{\infty} := \Big\{\forall\, i,\ Q_{\alpha\delta,i} \text{ is concentric cube to } Q_{\delta,i} \text{ with side length } \alpha\delta\Big\}. \tag{3.11}$$

We further define

$$\Omega_\delta := \bigcup_{i\in\mathbb{N}:\, Q_{\delta,i}\subseteq\Omega} Q_{\delta,i}, \qquad \Omega_{\alpha\delta} := \bigcup_{i\in\mathbb{N}:\, Q_{\delta,i}\subseteq\Omega} Q_{\alpha\delta,i}. \tag{3.12}$$

We now show that we may select $\alpha, \delta > 0$ such that

$$\max_{k=0,1,\dots,p}\left\{\sum_{q=k+1}\frac{1}{(q-k)!}\left\|U^{q,\varepsilon,\underline{m}}\right\|_{L^\infty(\Omega)}\delta^{q-k}\right\} \leq \varepsilon \tag{3.13}$$

$$\omega_{\varepsilon,\underline{m}}(\delta) \leq \varepsilon, \tag{3.14}$$

$$\big|\Omega\setminus\Omega_{\alpha\delta}\big| \leq \varepsilon. \tag{3.15}$$

Indeed, since $\omega_{\varepsilon,\underline{m}}$ is increasing and in view of (3.10)-(3.12), by choosing δ small enough we immediately obtain (3.13)-(3.14) and also

$$\big|\Omega\setminus\Omega_{\delta}\big| \leq \varepsilon. \tag{3.16}$$

In view of the identity

$$|\Omega_{\alpha\delta}| = \alpha^n\,|\Omega_\delta|, \tag{3.17}$$

by choosing

$$\alpha := \left(\max\left\{1-\frac{\varepsilon}{2|\Omega|},\, \frac{1}{2}\right\}\right)^{1/n}, \tag{3.18}$$

we obtain

$$\begin{aligned}
\big|\Omega\setminus\Omega_{\alpha\delta}\big| &= \big|\Omega\setminus\Omega_{\delta}\big| + \big|\Omega_\delta\setminus\Omega_{\alpha\delta}\big| \\
&\leq \varepsilon/2 + |\Omega_\delta| - \alpha^n|\Omega_\delta| \\
&\leq \varepsilon/2 + (1-\alpha^n)|\Omega| \\
&\leq \varepsilon
\end{aligned}$$

and (3.15) has been established as well. For each $i \in \mathbb{N}$ such that $Q_{\delta,i} \subseteq \Omega$, we consider a cut off function $\zeta^{\delta,i} \in C^\infty_c(Q_{\delta,i})$ such that

$$\chi_{Q_{\alpha\delta,i}} \leq \zeta^{\delta,i} \leq \chi_{Q_{\delta,i}} \tag{3.19}$$

and let

$$\{x^{\delta,i}\}_1^\infty := \Big\{\text{the centres of the cubes } \{Q_{\delta,i}\}_{i=1}^\infty\Big\} \subseteq \mathbb{R}^n.$$

We now define a mapping

$$u^{\varepsilon,\underline{m}} \in C^\infty_c(\Omega,\mathbb{R}^N)$$

which is given by

$$u^{\varepsilon,\underline{m}}(x) := \sum_{i\in\mathbb{N}:\, Q_{\delta,i}\subseteq\Omega}\zeta^{\delta,i}(x)\left(\sum_{q=0}^{p}\frac{1}{q!}U^{q,\varepsilon,\underline{m}}\big(x^{\delta,i}\big) : \big(x-x^{\delta,i}\big)^{\otimes q}\right). \tag{3.20}$$

Then, by (3.19), on each $Q_{\alpha\delta,i} \subseteq \Omega$, the map $u^{\varepsilon,\underline{m}}$ equals the restriction of a pth order polynomial and hence for any $k \in \{1,...,p\}$, we have

$$D^k u^{\varepsilon,\underline{m}}(x) \,=\, \sum_{q=k}^{p} \frac{1}{(q-k)!}\, U^{q,\varepsilon,\underline{m}}\big(x^{\delta,i}\big) : \big(x - x^{\delta,i}\big)^{\otimes(q-k)},$$

for $x \in Q_{\alpha\delta,i}$. By recalling the properties of the measurable set $F_{\varepsilon,\underline{m}} \subseteq \Omega$ of (3.3), for any $k \in \{0,1,...,p\}$ and for a.e. $x \in Q_{\alpha\delta,i} \setminus F_{\varepsilon,\underline{m}}$, we have

$$\begin{aligned}\Big|D^{k,h_{\underline{m}}}u \,-\, D^k u^{\varepsilon,\underline{m}}\Big|(x) \,&\leq\, \Big|D^{k,h_{\underline{m}}}u \,-\, U^{k,\varepsilon,\underline{m}}\Big|(x) \,+\, \Big|U^{k,\varepsilon,\underline{m}} \,-\, D^k u^{\varepsilon,\underline{m}}\Big|(x)\\ &\leq\, \varepsilon \,+\, \Bigg|U^{k,\varepsilon,\underline{m}}(x) \,-\\ &\qquad -\, \sum_{q=k}^{p} \frac{1}{(q-k)!}\, U^{q,\varepsilon,\underline{m}}\big(x^{\delta,i}\big) : \big(x - x^{\delta,i}\big)^{\otimes(q-k)}\Bigg|\end{aligned}$$

which by (3.9) gives that

$$\begin{aligned}\Big|D^k u^{\varepsilon,\underline{m}}(x) \,-\, D^{k,h_{\underline{m}}}u(x)\Big| \,&\leq\, \varepsilon \,+\, \Big|U^{k,\varepsilon,\underline{m}}(x) \,-\, U^{k,\varepsilon,\underline{m}}\big(x^{\delta,i}\big)\Big|\\ &\qquad +\, \sum_{q=k+1}^{p} \frac{1}{(q-k)!}\, \Big\|U^{q,\varepsilon,\underline{m}}\Big\|_{L^\infty(\Omega)} \big|x - x^{\delta,i}\big|^{q-k}\\ &\leq\, \varepsilon \,+\, \omega_{\varepsilon,\underline{m}}\big(\big|x - x^{\delta,i}\big|\big)\\ &\qquad +\, \sum_{q=k+1}^{p} \frac{1}{(q-k)!}\, \Big\|U^{q,\varepsilon,\underline{m}}\Big\|_{L^\infty(\Omega)}\, \delta^{q-k}\end{aligned}$$

for a.e. $x \in Q_{\alpha\delta,i} \setminus F_{\varepsilon,\underline{m}}$. Hence,

$$\begin{aligned}\Big|D^k u^{\varepsilon,\underline{m}} \,-\, D^{k,h_{\underline{m}}}u\Big| \,&\leq \varepsilon \,+\, \omega_{\varepsilon,\underline{m}}(\delta)\\ &\qquad +\, \sum_{q=k+1}^{p} \frac{1}{(q-k)!}\, \Big\|U^{q,\varepsilon,\underline{m}}\Big\|_{L^\infty(\Omega)}\, \delta^{q-k},\end{aligned} \tag{3.21}$$

a.e. on $\in Q_{\alpha\delta,i} \setminus F_{\varepsilon,\underline{m}}$. By (3.21) and (3.10)-(3.14) we deduce that

$$\max_{k=0,1,...,p} \Big\|D^k u^{\varepsilon,\underline{m}} \,-\, D^{k,h_{\underline{m}}}u\Big\|_{L^\infty(\Omega_{\alpha\delta}\setminus F_{\varepsilon,\underline{m}})} \,\leq\, 3\varepsilon. \tag{3.22}$$

Finally, we set

$$E_{\varepsilon,\underline{m}} \,:=\, F_{\varepsilon,\underline{m}} \cup \big(\Omega \setminus \Omega_{\alpha\delta}\big). \tag{3.23}$$

Then, by (3.23), (3.15) and (3.3) we have that

$$\big|E_{\varepsilon,\underline{m}}\big| \,\leq\, 2\varepsilon$$

and (3.22), (3.23) imply

$$\left\{\begin{aligned}\Big\|D^{[p]}u^{\varepsilon,\underline{m}} \,-\, D^{[p],h_{\underline{m}}}u\Big\|_{L^\infty(\Omega\setminus E_{\varepsilon,\underline{m}})} \,&\leq\, 3p\,\varepsilon,\\ \big\|u^{\varepsilon,\underline{m}} \,-\, u\big\|_{L^\infty(\Omega\setminus E_{\varepsilon,\underline{m}})} \,&\leq\, 3\varepsilon.\end{aligned}\right. \tag{3.24}$$

Hence, by replacing ε by $\varepsilon/3p$, we see that (3.1) has been established.

Step 4. We now establish (3.2) under the additional hypothesis that $u \in L^r(\Omega, \mathbb{R}^N)$. We begin by noting that on top of (3.3) we can also arrange to have

$$\big\| u \,-\, U^{0,\varepsilon,\underline{m}} \big\|_{L^r(\Omega)} \,\leq\, \varepsilon. \tag{3.25}$$

This follows by the next simple modification of Step 2: we replace $T^R(V)$ by $\big(u, T^R\big(D^{[p],h_{\underline{m}}}u\big)\big)$, choose $s := r$ and use that u can be approximated in the L^r norm by smooth compactly supported mappings. Then we obtain (3.25) and the first and last inequalities of (3.3). The middle inequality of (3.3) follows by (3.25) and (by perhaps modifying $F_{\varepsilon,\underline{m}}$ and the choice of ε accordingly):

$$\begin{aligned}\Big|\Big\{\big|u - U^{0,\varepsilon,\underline{m}}\big| > \varepsilon^{1/2}\Big\}\Big| \,&\leq\, \varepsilon^{-r/2}\int_\Omega \big|u \,-\, U^{0,\varepsilon,\underline{m}}\big|^r \\ &\leq\, \varepsilon^{r/2}.\end{aligned}$$

Next, by (3.25) and by (3.20) we have that

$$\begin{aligned}\big\|u - u^{\varepsilon,\underline{m}}\big\|^r_{L^r(\Omega)} \,&\leq\, 2^r\varepsilon \,+\, 2^r\int_\Omega \Big|u^{\varepsilon,\underline{m}} \,-\, U^{0,\varepsilon,\underline{m}}\Big|^r \\ &\leq\, 2^r\varepsilon \,+\, 2^r\big|\Omega\setminus\Omega_\delta\big|\,\big\|U^{0,\varepsilon,\underline{m}}\big\|^r_{L^\infty(\Omega)} \\ &\qquad +\, 4^r\int_{\Omega_\delta\setminus\Omega_{\alpha\delta}}\Big\{\big|u^{\varepsilon,\underline{m}}\big|^r + \big|U^{0,\varepsilon,\underline{m}}\big|^r\Big\} \\ &\qquad +\, 2^r\int_{\Omega_{\alpha\delta}}\Big|u^{\varepsilon,\underline{m}} \,-\, U^{0,\varepsilon,\underline{m}}\Big|^r\end{aligned}$$

which gives

$$\begin{aligned}\big\|u - u^{\varepsilon,\underline{m}}\big\|^r_{L^r(\Omega)} \,&\leq\, 4^r\varepsilon \,+\, 4^r\big|\Omega\setminus\Omega_\delta\big|\,\big\|U^{0,\varepsilon,\underline{m}}\big\|^r_{L^\infty(\Omega)} \\ &\qquad +\, 4^r\sum_{i\in\mathbb{N}:\, Q_{\delta,i}\subseteq\Omega}\Bigg\{\int_{Q_{\delta,i}\setminus Q_{\alpha\delta,i}}\Big\{\big|u^{\varepsilon,\underline{m}}\big|^r + \big|U^{0,\varepsilon,\underline{m}}\big|^r\Big\} \\ &\qquad\qquad +\int_{Q_{\alpha\delta,i}}\Big|u^{\varepsilon,\underline{m}} \,-\, U^{0,\varepsilon,\underline{m}}\Big|^r\Bigg\}.\end{aligned} \tag{3.26}$$

Moreover, by (3.20) (and (3.19)) we have the estimates

$$\left\{\begin{aligned} \big|u^{\varepsilon,\underline{m}}\big| \,&\leq\, \sum_{q=0}^{p}\frac{1}{q!}\,\big\|U^{q,\varepsilon,\underline{m}}\big\|_{L^\infty(\Omega)}\,\delta^q, \qquad \text{on } Q_{\delta,i}\setminus Q_{\alpha\delta,i},\ i\in\mathbb{N}, \\ \big|u^{\varepsilon,\underline{m}} \,-\, U^{0,\varepsilon,\underline{m}}\big| \,&\leq\, \sum_{q=1}^{p}\frac{1}{q!}\,\big\|U^{q,\varepsilon,\underline{m}}\big\|_{L^\infty(\Omega)}\,\delta^q, \qquad \text{on } Q_{\alpha\delta,i},\ i\in\mathbb{N}.\end{aligned}\right. \tag{3.27}$$

By inserting (3.27) into (3.26) we obtain the estimate

$$\begin{aligned}\big\|u \,-\, u^{\varepsilon,\underline{m}}\big\|^r_{L^r(\Omega)} \,&\leq\, 4^r\varepsilon \,+\, 4^r\big|\Omega\setminus\Omega_\delta\big|\,\big\|U^{0,\varepsilon,\underline{m}}\big\|^r_{L^\infty(\Omega)} \\ &\qquad +\, 6^r\big|\Omega_\delta\setminus\Omega_{\alpha\delta}\big|\left(\sum_{q=0}^{p}\frac{1}{q!}\,\big\|U^{q,\varepsilon,\underline{m}}\big\|_{L^\infty(\Omega)}\,\delta^q\right)^r \\ &\qquad +\, 4^r|\Omega|\left(\sum_{q=1}^{p}\frac{1}{q!}\,\big\|U^{q,\varepsilon,\underline{m}}\big\|_{L^\infty(\Omega)}\,\delta^q\right)^r\end{aligned}$$

which in turn gives

$$
\begin{aligned}
\big\|u\,-\,u^{\varepsilon,\underline{m}}\big\|_{L^r(\Omega)}\,\leq\,4\varepsilon\,+\,4\big|\Omega\setminus\Omega_\delta\big|^{1/r}\,\big\|U^{0,\varepsilon,\underline{m}}\big\|_{L^\infty(\Omega)}& \\
+\,6(1-\alpha^n)^{1/r}\left(|\Omega|^{1/r}\sum_{q=0}^{p}\frac{1}{q!}\,\big\|U^{q,\varepsilon,\underline{m}}\big\|_{L^\infty(\Omega)}\right)& \\
+\,4|\Omega|\sum_{q=1}^{p}\frac{1}{q!}\,\big\|U^{q,\varepsilon,\underline{m}}\big\|_{L^\infty(\Omega)}\,\delta^q.&
\end{aligned}
\tag{3.28}
$$

In view of (3.28) and of (3.13)-(3.18), by decreasing δ further and by increasing α even further, we can achieve

$$
\big\|u\,-\,u^{\varepsilon,\underline{m}}\big\|_{L^r(\Omega)}\,\leq\,7\varepsilon.
$$

Hence, the desired conclusion follows by replacing ε by $\varepsilon/7$. The lemma has been established. $\square$

By utilising Lemma 10, we may now state and prove the main result of this paper.

Theorem 11 (Mollification of $\mathcal{D}$-solutions)**.** *Let $\Omega\subseteq\mathbb{R}^n$ be an open set and*

$$
F\ :\ \Omega\times\Big(\mathbb{R}^N\times\mathbb{R}^{Nn}\times\cdots\times\mathbb{R}^{Nn^p}_s\Big)\longrightarrow\mathbb{R}^M
$$

a Carathéodory map. Suppose that $u:\Omega\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^N$ is a $\mathcal{D}$-solution to the pth order PDE system

$$
F\Big(x,u(x),D^{[p]}u(x)\Big)\,=\,0,\qquad x\in\Omega.
$$

Then, there exists a multi-indexed sequence of maps $(u^{\underline{m}})_{\underline{m}\in\mathbb{N}^{p\times p}}\subseteq C^\infty_0(\Omega,\mathbb{R}^N)$ with the following properties: for any diffuse pth order jet $\mathcal{D}^{[p]}u$ generated along subsequences of an infinitesimal multi-indexed sequence $(h_{\underline{m}})_{\underline{m}\in\mathbb{N}^{p\times p}}$, there exists a single-indexed subsequence

$$
(u^\nu)_{\nu\in\mathbb{N}}\,\subseteq\,(u^{\underline{m}})_{\underline{m}\in\mathbb{N}^{p\times p}}
\tag{3.29}
$$

with the following properties:

$$
\left\{
\begin{array}{lll}
u^\nu\longrightarrow u & & a.e.\ on\ \Omega,\\[1ex]
\delta_{D^{[p]}u^\nu}\overset{*}{-\!\!\!\rightharpoonup}\mathcal{D}^{[p]}u & & in\ \mathscr{Y}\Big(\Omega,\overline{\mathbb{R}}^{Nn}\times\ldots\times\overline{\mathbb{R}}^{Nn^p}_s\Big),
\end{array}
\right.
\tag{3.30}
$$

both as $\nu\to\infty$. In addition, for each $\nu\in\mathbb{N}$, the map $u^\nu\in C^\infty_0(\Omega,\mathbb{R}^N)$ is a smooth strong solution to the approximate pth order PDE system

$$
F\Big(x,u^\nu(x),D^{[p]}u^\nu(x)\Big)\,=\,f^\nu(x),\qquad x\in\Omega
\tag{3.31}
$$

and $f^\nu:\Omega\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^M$ is a measurable map satisfying $f^\nu\longrightarrow 0$ as $\nu\to\infty$ in the following sense:

$$
\textit{For any } \Phi\in C^0_c\big(\mathbb{R}^{Nn}\times\cdots\times\mathbb{R}^{Nn^p}_s\big),\quad \Phi\big(D^{[p]}u^\nu\big)f^\nu\longrightarrow 0,\ \ a.e.\ on\ \Omega.
\tag{3.32}
$$

Note that the second statement of (3.30) is interesting because the (fibre) product Young measure $\mathcal{D}^{[p]}u$ can be weakly* approximated by the product Young measures $\delta_{D^{[p]}u^\nu}$ as $\nu\to\infty$ (confer with Definition 6). The following consequence of our constructions will also follow from Theorem 11.

Corollary 12 (Mollification of $\mathcal{D}$-solutions contn'd)**.** *In the setting of Theorem 11, the mode of convergence (3.32) is (up to the passage to a subsequence) equivalent to either of the modes of convergence as $\nu \to \infty$:*

$$\left\{ \begin{array}{l} (a)\ \text{For any } R>0, \quad \chi_{\mathbb{B}_R(0)}\big(D^{[p]}u^\nu\big) f^\nu \longrightarrow 0, \ \text{a.e. on } \Omega. \\ (b)\ \text{For any } \varepsilon>0, \ \text{and any } E\subseteq\Omega \text{ with } |E|<\infty, \\ \qquad\qquad \Big| E \cap \big\{|f^\nu|>\varepsilon\big\} \cap \big\{\big|D^{[p]}u^\nu\big|<1/\varepsilon\big\}\Big| \longrightarrow 0. \end{array} \right. \tag{3.33}$$

In addition, we have

$$f^\nu \longrightarrow 0 \qquad \text{a.e. on } \Omega\setminus E, \tag{3.34}$$

where

$$E := \bigcup_{q=1}^{p} \Big\{ x\in\Omega \,:\, \big[\mathcal{D}^q u(x)\big]\big(\{\infty\}\big) > 0 \Big\}. \tag{3.35}$$

The next example shows that in general it is not possible to strengthen the modes of convergence in (3.32)-(3.35) to L^p or even to a.e. on Ω even for linear equations:

Example 13 (Optimality of Theorem 11)**.** *Consider the equation*

$$u' = 0, \quad u : (0,1)\subseteq\mathbb{R}\to\mathbb{R}.$$

(I) Let $u_c \in C^0_0(0,1)$ be a Cantor-type function[2] *with $u_c \neq 0$ and $u_c' = 0$ a.e. on $(0,1)$. Then, u_c is a $\mathcal{D}$-solution because it satisfies the equation a.e. on $(0,1)$. However, for any hypothetical approximate equation*

$$u_c^{\varepsilon\,\prime} = f^\varepsilon, \quad \text{on } (0,1)$$

such that $u_c^\varepsilon \longrightarrow u_c$ a.e. on Ω and $f^\varepsilon \longrightarrow 0$ in $L^1(0,1)$, by Poincaré inequality we have

$$\|u_c^\varepsilon\|_{L^1(0,1)} \leq C\|u_c^{\varepsilon\,\prime}\|_{L^1(0,1)} = C\|f^\varepsilon\|_{L^1(0,1)} \longrightarrow 0, \quad \text{as } \varepsilon\to 0,$$

while $u_c^\varepsilon \longrightarrow u_c$ a.e. on Ω and $u_c \neq 0$.

(II) Let $u_s \in L^1_c(0,1)$ be a singular solution to the equation such that for any diffuse derivative $\mathcal{D}u_s \in \mathscr{Y}\big((0,1),\overline{\mathbb{R}}\big)$, we have $\mathcal{D}u_s \equiv \delta_{\{\infty\}}$ on a set of positive measure. Such a solution can be constructed by taking a compact nowhere dense set $K\subseteq(0,1)$ of positive measure (e.g., we may take

$$K = [1/3,\,2/3]\setminus \cup_{j=1}^{\infty}\big(q_j - 3^{-2j}, q_j + 3^{-2j}\big)$$

where $(q_j)_1^\infty$ is an enumeration of the rationals in $[1/3,\,2/3]$) and setting $u_s := \chi_K$. Then, u_s satisfies

$$|D^{1,h}u_s(x)| \longrightarrow 0 \text{ for } x\in(0,1)\setminus K, \quad |D^{1,h}u_s(x)| \longrightarrow \infty \text{ for } x\in K,$$

as $h\to 0$. By Lemma 3, we have that all diffuse gradients coincide and are given by

$$\mathcal{D}u_s(x) = \chi_{(0,1)\setminus K}(x)\,\delta_{\{0\}} + \chi_K(x)\,\delta_{\{\infty\}}, \qquad \text{a.e. } x\in(0,1).$$

[2]The reader should not be alarmed by the fact that $\mathcal{D}$-solutions satisfy such "unpleasant" nonuniqueness properties. On the one hand, this behaviour is a common feature of all "a.e.-type" notions of solution, e.g. L^p-viscosity solutions, piecewise solutions, etc. On the other hand, in the vectorial nonlinear case, most interesting systems present such issues even for smooth C^∞ solutions (e.g. the minimal surface system in codimension greater than 1, see [LO]). **However**, by imposing extra constraints on top of the boundary conditions (which depend on the nature of the problem), **$\mathcal{D}$-solutions are indeed unique** and pathological solutions as in this example are ruled out, see [K8] and [K10].

This implies that u_s is a $\mathcal{D}$-solution: indeed, we have that either supp$_*(\mathcal{D}u_s(x)) = \{0\}$ or supp$_*(\mathcal{D}u_s(x)) = \emptyset$ and hence

$$\text{for a.e. } x \in (0,1) \text{ and all } X_x \in \text{supp}_*(\mathcal{D}u_s(x)) \implies |X_x| = 0.$$

However, it is impossible to obtain $f^\varepsilon \longrightarrow 0$ a.e. on $(0,1)$ for any approximation scheme as in the theorem such that

$$u_s^{\varepsilon\prime} = f^\varepsilon \quad \text{on } (0,1), \quad u_s^\varepsilon \longrightarrow u_c \text{ a.e. on } \Omega.$$

Indeed, by (3.30) we would have $\delta_{u_s^{\varepsilon\prime}} \overset{*}{\rightharpoonup} \mathcal{D}u_s$ in $\mathscr{Y}\big((0,1),\overline{\mathbb{R}}\big)$ as $\varepsilon \to 0$ and hence we have $\delta_{u_s^{\varepsilon\prime}} \overset{*}{\rightharpoonup} \delta_{\{\infty\}}$ in $\mathscr{Y}\big(K,\overline{\mathbb{R}}\big)$. Hence, by Lemma 3 this would give $|f^\varepsilon(x)| = |u_s^{\varepsilon\prime}(x)| \longrightarrow \infty$ for a.e. $x \in K$.

For the proof of Theorem 11 we need three lemmas which are given right next. The first two are variants of results established in [K8], while the third one is a consequence of standard results on Young measures. They are all given below in the generality of Young measures because they do not utilise the special structure of diffuse derivatives. For the sake of completeness, we provide the first two results in full by giving all the details of their proofs, while for the third we give a precise reference.

Lemma 14 (Convergence lemma V2, cf. [K8])**.** *Suppose that $u^m \longrightarrow u^\infty$ a.e. on Ω, as $m \to \infty$ where $u^\infty, (u^m)_1^\infty$ are measurable maps $\Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$. Let $\mathbb{W}$ be a finite dimensional metric vector space, isometrically and densely contained into a compactification $\mathbb{K}$ of itself. Suppose also we are given Carathéodory mappings*

$$F^\infty, F^m \ : \ \Omega \times \big(\mathbb{R}^N \times \mathbb{W}\big) \longrightarrow \mathbb{R}^M, \quad m \in \mathbb{N},$$

and we are also given Young measures $\vartheta^\infty, (\vartheta)_1^\infty$ in $\mathscr{Y}\big(\Omega, \mathbb{K}\big)$ such that the following modes of convergence hold true:

$$F^\mu(x,\cdot,\cdot) \longrightarrow F^\infty(x,\cdot,\cdot) \quad \text{in } C^0(\mathbb{R}^N \times \mathbb{W}), \quad \text{as } m \to \infty, \quad \text{a.e. } x \in \Omega,$$
$$\vartheta^m \overset{*}{\rightharpoonup} \vartheta^\infty \quad \text{in } \ \mathscr{Y}\big(\Omega, \mathbb{K}\big), \quad \text{as } m \to \infty.$$

Then, if for a given function $\Phi \in C^0_c(\mathbb{W})$ we have

$$\int_{\mathbb{K}} \big|\Phi(\boldsymbol{X})\, F^\infty\big(x, u^\infty(x), \boldsymbol{X}\big)\big|\, d[\vartheta^\infty(x)](\boldsymbol{X}) \,=\, 0, \quad \text{a.e. } x \in \Omega,$$

it follows that

$$\lim_{m\to\infty} \int_{\mathbb{K}} \big|\Phi(\boldsymbol{X})\, F^m\big(x, u^m(x), \boldsymbol{X}\big)\big|\, d[\vartheta^m(x)](\boldsymbol{X}) \,=\, 0, \quad \text{a.e. } x \in \Omega.$$

We will later apply this lemma to the case of $\mathbb{W} = \mathbb{R}^{Nn} \times \cdots \times \mathbb{R}_s^{Nn^p}$ for the compactification of $\mathbb{W}$ given by the torus $\mathbb{K} = \overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}_s^{Nn^p}$. We remind that the metric on $\mathbb{W}$ is the product metric induced by the imbedding of $\mathbb{R}_s^{Nn^q}$ into $\overline{\mathbb{R}}_s^{Nn^p}$, $q = 1, ..., p$.

Proof of Lemma 14. We first fix $\Phi \in C^0_c(\mathbb{W})$ and define

$$\phi^m(x) \,:=\, \Big\| \Phi(\cdot)\, F^m\big(x, u^m(x), \cdot\big) \,-\, \Phi(\cdot)\, F^\infty\big(x, u^\infty(x), \cdot\big) \Big\|_{C^0(\mathbb{W})}$$

and we claim that our convergence hypotheses imply $\phi^m \longrightarrow 0$ a.e. on Ω. Indeed, let us fix $x \in \Omega$ such that $u^m(x) \longrightarrow u^\infty(x)$ (and the set of such points x has full measure). Then, we can find compact sets $K' \Subset \mathbb{R}^N$ and $K'' \Subset \mathbb{W}$ such that

for large $m \in \mathbb{N}$ we have $u^m(x), u^\infty(x) \in K'$ and also $\mathrm{supp}(\Phi) \subseteq K''$. By the convergence assumption on the maps F^m, we have

$$\big\|F^m(x,\cdot) - F^m(x,\cdot)\big\|_{C^0(K'\times K'')} \longrightarrow 0, \quad \text{as } m \to \infty.$$

If $\omega_x^\infty \in C^0[0,\infty)$ is the modulus of continuity of the map $K' \ni \xi \mapsto F^\infty(x,\xi,\mathbf{X}) \in \mathbb{R}^M$ (that is $\omega_x^\infty \geq \omega_x^\infty(0) = 0$) which is uniform with respect to $\mathbf{X} \in K''$, we have

$$\begin{aligned}
|\phi^m(x)| \,\leq\, \|\Phi\|_{C^0(K'')} & \left\{ \Big\| F^\infty\big(x, u^m(x), \cdot\big) \,-\, F^\infty\big(x, u^\infty(x), \cdot\big)\Big\|_{C^0(K'')} \right. \\
& \left. +\, \Big\| F^m\big(x, u^m(x), \cdot\big) \,-\, F^\infty\big(x, u^m(x), \cdot\big)\Big\|_{C^0(K'')} \right\} \\
\leq\, \|\Phi\|_{C^0(\mathbb{W})} & \Big\{ \,\omega_x^\infty\big(\big|u^m(x) - u^\infty(x)\big|\big) \,+\, \big\|F^m(x,\cdot) - F^\infty(x,\cdot)\big\|_{C^0(K'\times K'')} \Big\} \\
\longrightarrow 0, &
\end{aligned}$$

as $m \to \infty$. Since this happens for a set of points $x \in \Omega$ of full measure, we deduce that $\phi^m(x) \longrightarrow 0$ for a.e. $x \in \Omega$. We now fix $R > 0$ and $\Phi \in C_c^0(\mathbb{W})$ as in the statement of the lemma and set

$$\Omega_R \,:=\, \mathbb{B}_R(0) \cap \Big\{ x \in \Omega \,:\, \big\|\Phi(\cdot)\, F^\infty\big(x, u^\infty(x), \cdot\big)\big\|_{C^0(\mathbb{W})} <\, R \Big\}.$$

Since $|\Omega_R| < \infty$, by Egoroff's theorem we can find a measurable sequence $\{E_i\}_1^\infty \subseteq \Omega_R$ such that $|E_i| \to 0$ as $i \to \infty$ and for each $i \in \mathbb{N}$ fixed,

$$\|\phi^m\|_{L^\infty(\Omega_R\setminus E_i)} \longrightarrow 0, \quad \text{as } m \to \infty.$$

Since $|\Omega_R| < \infty$, this gives $\phi^m \longrightarrow 0$ in $L^1(\Omega_R \setminus E_i)$ as $m \to \infty$. This convergence and the form of Ω_R imply that the Carathéodory functions

$$\begin{aligned}
\Psi^m(x,\mathbf{X}) \,&:=\, \big|\Phi(\mathbf{X})\, F^m\big(x, u^m(x), \mathbf{X}\big)\big|\,, \\
\Psi^\infty(x,\mathbf{X}) \,&:=\, \big|\Phi(\mathbf{X})\, F^\infty\big(x, u^\infty(x), \mathbf{X}\big)\big|\,,
\end{aligned}$$

are elements of the space $L^1\big(\Omega_R \setminus E_i, C^0(\mathbb{K})\big)$ because

$$\|\phi^m\|_{L^1(\Omega_R\setminus E_i)} \,\geq\, \|\Psi^m - \Psi\|_{L^1(\Omega_R\setminus E_i, C^0(\mathbb{K}))}$$

and for m large we have

$$\|\Psi^m\|_{L^1(\Omega_R\setminus E_i)} \,\leq\, 1 \,+\, \|\Psi^\infty\|_{L^1(\Omega_R\setminus E_i)} \,\leq\, 1 + \,|\Omega_R|\, R.$$

Moreover, we have

$$\Psi^m \longrightarrow \Psi^\infty \quad \text{in } L^1\big(\Omega_R \setminus E_i, C^0(\mathbb{K})\big), \quad \text{as } m \to \infty$$

and also by assumption $\vartheta^m \overset{*}{\rightharpoonup} \vartheta^\infty$ in $\mathscr{Y}\big(\Omega_R \setminus E_i, \mathbb{K}\big)$ as $m \to \infty$. Thus, the weak*-strong continuity of the duality pairing

$$L^\infty_{w^*}\Big(\Omega_R \setminus E_i, \mathcal{M}(\mathbb{K})\Big) \,\times\, L^1\Big(\Omega_R \setminus E_i, C^0(\mathbb{K})\Big) \longrightarrow \mathbb{R},$$

implies

$$\begin{aligned}
\lim_{m\to\infty} \int_{\Omega_R\setminus E_i} \int_{\mathbb{K}} & \big|\Phi(\mathbf{X})\, F^m\big(x, u^m(x), \mathbf{X}\big)\big|\, d[\vartheta^m(x)](\mathbf{X})\, dx \\
&= \int_{\Omega_R\setminus E_i} \int_{\mathbb{K}} \big|\Phi(\mathbf{X})\, F^\infty\big(x, u^\infty(x), \mathbf{X}\big)\big|\, d[\vartheta^\infty(x)](\mathbf{X})\, dx.
\end{aligned}$$

We recall now that by our hypothesis the right hand side of the above vanishes in order to obtain the desired conclusion after letting $i \to \infty$ and then taking $R \to \infty$. $\square$

The next lemma says that if the distance between two sequences of measurable maps asymptotically vanishes, then the maps represent the same Young measure in the compactification (see also [K8, FL]).

Lemma 15. *Let $\mathbb{W}$ be a finite dimensional metric vector space, isometrically and densely contained into a compactification $\mathbb{K}$ of itself. Let also $E \subseteq \mathbb{R}^n$ be a measurable set. If $U^m, V^m : E \subseteq \mathbb{R}^n \longrightarrow \mathbb{W}$ are measurable maps satisfying*

$$\delta_{U^m} \overset{*}{\rightharpoonup} \vartheta \ \text{ in } \mathscr{Y}(E,\mathbb{K}), \qquad |U^m - V^m| \longrightarrow 0 \ \text{ a.e. on } E,$$

as $m \to \infty$, then $\delta_{V^m} \overset{*}{\rightharpoonup} \vartheta$ in $\mathscr{Y}(E,\mathbb{K})$, as $m \to \infty$.

Proof of Lemma 15. We begin by fixing $\varepsilon > 0$, $\phi \in L^1(E)$ and $\Phi \in C^0(\mathbb{K})$. Since Φ is uniformly continuous on $\mathbb{K}$, there exists a bounded increasing modulus of continuity $\omega \in C^0[0,\infty)$ such that

$$|\Phi(X) - \Phi(Y)| \,\leq\, \omega(|X-Y|), \qquad \text{for } X, Y \in \mathbb{K}$$

and we also have $\|\omega\|_{C^0(0,\infty)} < \infty$. Moreover, since $|U^m - V^m| \longrightarrow 0$ a.e. on E, we have that $|U^m - V^m| \longrightarrow 0$ μ-a.e. on E where μ is the absolutely continuous finite measure given by $\mu(A) := \|\phi\|_{L^1(A)}$, $A \subseteq E$. It follows that $|U^m - V^m| \longrightarrow 0$ in μ-measure as well. As a consequence, we obtain

$$\begin{aligned}\left|\int_E \phi\Big[\Phi(V^m) - \Phi(U^m)\Big]\right| \,&\leq\, \int_E |\phi|\, \omega\big(|U^m - V^m|\big)\\ &\leq\, \|\omega\|_{C^0(0,\infty)}\, \mu\big(\{|U^m - V^m| > \varepsilon\}\big) \,+\, \omega(\varepsilon)\, \mu(E).\end{aligned}$$

By letting first $m \to \infty$ and then $\varepsilon \to 0$, the density in $L^1\big(E, C^0(\mathbb{K})\big)$ of the linear span of the products of the form $\phi(x)\Phi(X)$ implies the desired conclusion. $\square$

The following result is the last ingredient needed for the proof of Theorem 11.

Lemma 16. *Let $\mathbb{W}'$ and $\mathbb{W}''$ be finite dimensional metric vector spaces, isometrically and densely contained into certain compactification $\mathbb{K}'$ and $\mathbb{K}''$ of $\mathbb{W}'$, $\mathbb{W}''$ respectively. Let also $E \subseteq \mathbb{R}^n$ be a measurable set. If $U^m : E \subseteq \mathbb{R}^n \longrightarrow \mathbb{W}'$, $V^m : E \subseteq \mathbb{R}^n \longrightarrow \mathbb{W}''$ are sequences of measurable maps satisfying*

$$U^m \longrightarrow U \ \text{ a.e. on } E, \qquad \delta_{V^m} \overset{*}{\rightharpoonup} \vartheta \ \text{ in } \mathscr{Y}(E,\mathbb{K}''),$$

as $m \to \infty$, then

$$\delta_{(U^m,V^m)} \overset{*}{\rightharpoonup} \delta_U \times \vartheta \quad \text{in } \mathscr{Y}(E,\, \mathbb{K}' \times \mathbb{K}''), \ \text{ as } m \to \infty.$$

Proof of Lemma 16. The proof of this result can be found (actually in a much more general topological setting) e.g. in [FG], Corollary 3.89 on p. 257. $\square$

Now we may prove our main result.

Proof of Theorem 11. Step 1. We begin by noting a general convergence fact. Let $(\mathfrak{X}, \rho)$ be a metric space, $f \in \mathfrak{X}$ and let also $\underline{m} \in \mathbb{N}^{p\times p}$ be a matrix of indices as in Definition 4. Suppose that $\{f^{\underline{m}}\}_{\underline{m}\in\mathbb{N}^{p\times p}} \subseteq \mathfrak{X}$ is a multi-indexed sequence such that the successive limit converges to f in $\mathfrak{X}$ (Definition 5):

$$\lim_{\underline{m}\to\infty} \rho\,(f^{\underline{m}},\, f) \,=\, \lim_{m_p^p\to\infty}\left(\dots\dots \lim_{m_2^2\to\infty}\left(\lim_{m_2^1\to\infty}\left(\lim_{m_1^1\to\infty} \rho\,(f^{\underline{m}},\, f)\right)\right)\right) \,=\, 0.$$

Then, there exist subsequences $(m^b_{a,\nu})_{\nu=1}^\infty$, $a,b\in\{1,...,p\}$, such that, if $(\underline{m}_\nu)_{\nu=1}^\infty\subseteq\mathbb{N}^{p\times p}$ is the single-indexed sequence with components $m^b_{a,\nu}$, then $f^{\underline{m}_\nu}$ converges to f in $\mathfrak{X}$:

$$\lim_{\nu\to\infty}\rho\left(f^{\underline{m}_\nu},\,f\right)\,=\,0.$$

This is a simple consequence of the definitions of limits.

Step 2. Let $u:\Omega\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^N$ be a $\mathcal{D}$-solution to the system

$$F\Big(x,u(x),D^{[p]}u(x)\Big)\,=\,0,\qquad x\in\Omega,$$

and let $\mathcal{D}^{[p]}u=\mathcal{D}u\times\cdots\mathcal{D}^pu$ be a pth order Jet of u arising along matrix-indexed infinitesimal subsequences

$$\delta_{D^{[p]},\underline{m}}u\overset{*}{\rightharpoonup}\mathcal{D}^{[p]}u\quad\text{ in }\mathscr{Y}\Big(\Omega,\overline{\mathbb{R}}^{Nn}\times...\times\overline{\mathbb{R}}^{Nn^p}_s\Big),$$

as $\underline{m}\to\infty$ (Definitions 4, 5, 6, 7). Since the weak* topology on the Young measures is metrisable, we may apply Step 1 to

$$\mathfrak{X}\,=\,\mathscr{Y}\Big(\Omega,\overline{\mathbb{R}}^{Nn}\times...\times\overline{\mathbb{R}}^{Nn^p}_s\Big),\qquad f^{\underline{m}}=\,\delta_{D^{[p]},\underline{m}}u,\qquad f\,=\,\mathcal{D}^{p]}u,$$

for some metric ρ inducing the weak* topology. Thus, we infer that there exists a single-indexed subsequence $(\underline{m}_\nu)_1^\infty\subseteq\mathbb{N}^{p\times p}$ such that

$$\lim_{\nu\to\infty}\rho\left(\delta_{D^{[p]},\underline{m}_\nu}u,\,\mathcal{D}^{p]}u\right)\,=\,0,$$

because by the definition of $\mathcal{D}^{[p]}u$ we have

$$\lim_{m^p_p\to\infty}\left(......\lim_{m^2_2\to\infty}\left(\lim_{m^1_2\to\infty}\left(\lim_{m^1_1\to\infty}\rho\left(\delta_{D^{[p]},\underline{m}}u,\,\mathcal{D}^{p]}u\right)\right)\right)\right)\,=\,0.$$

Since u is measurable, by invoking Lemma 10 for $\varepsilon=1/|\underline{m}|$ we obtain a multi-indexed sequence $(u^{\underline{m}})_{\underline{m}\in\mathbb{N}^{p\times p}}\subseteq C^\infty_0(\Omega,\mathbb{R}^N)$. Let $(u^\nu)_{\nu\in\mathbb{N}}$ be the subsequence of it corresponding to $(\underline{m}_\nu)_1^\infty$. Then, by (3.1) and by recalling that almost uniform implies a.e. convergence, we immediately have $u^\nu\longrightarrow u$ a.e. on Ω as $\nu\to\infty$. Further, again by (3.1) we have

$$\Big|D^{[p],\underline{m}_\nu}u\,-\,D^{[p]}u^\nu\Big|\,\longrightarrow 0\quad\text{ a.e. on }\Omega,\quad\text{as }\nu\to\infty.$$

By Lemma 15 and the above, we obtain

$$\delta_{D^{[p]}u^\nu}\overset{*}{\rightharpoonup}\mathcal{D}^{[p]}u\quad\text{ in }\mathscr{Y}\Big(\Omega,\overline{\mathbb{R}}^{Nn}\times...\times\overline{\mathbb{R}}^{Nn^p}_s\Big),\quad\text{as }\nu\to\infty.$$

Step 3. We now *define* for each $\nu\in\mathbb{N}$ the mapping $f^\nu:\Omega\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^M$ given by

$$f^\nu(x)\,:=\,F\Big(x,u^\nu(x),D^{[p]}u^\nu(x)\Big),\qquad x\in\Omega.$$

In order to conclude the theorem, we seek to show that for any

$$\Phi\,\in C^0_c\big(\mathbb{R}^{Nn}\times\cdots\times\mathbb{R}^{Nn^p}_s\big),$$

we have that

$$\text{for a.e. }x\in\Omega,\quad\Phi\Big(D^{[p]}u^\nu(x)\Big)f^\nu(x)\longrightarrow 0\ \text{ as }\nu\to\infty.$$

This last statement is a consequence of the Convergence Lemma 14. Indeed, since u is a $\mathcal{D}$-solution on Ω, for a.e. $x\in\Omega$, we have

$$\big|F\big(x,u(x),\underline{\mathbf{X}}\big)\big|\,=\,0,\qquad\underline{\mathbf{X}}\in\operatorname{supp}_*\big(\mathcal{D}^{[p]}u(x)\big).$$

We fix such an x as above and we choose a function Φ as above. Then, we note that the definition of $\mathcal{D}$-solutions implies that the continuous function

$$\overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s \ \ni \ \underline{\mathbf{X}} \longmapsto \big|\Phi(\underline{\mathbf{X}})\, F\big(x, u(x), \underline{\mathbf{X}}\big)\big| \ \in \ \mathbb{R}$$

is well-defined on the compactification and vanishes on the support of the probability measure $\mathcal{D}^{[p]}u(x)$. Hence, we have

$$\int_{\overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s} \big|\Phi(\underline{\mathbf{X}})\, F\big(x, u(x), \underline{\mathbf{X}}\big)\big|\, d\big[\mathcal{D}^{[p]}u(x)\big](\underline{\mathbf{X}}) \ = \ 0, \qquad \text{a.e. } x \in \Omega.$$

Since $u^\nu \longrightarrow u$ a.e. on Ω and $\delta_{D^{[p]}u^\nu} \overset{*}{\rightharpoondown} \mathcal{D}^{[p]}u$, by applying Lemma 14 we obtain that

$$\begin{aligned}
\Big|\Phi\big(D^{[p]}u^\nu\big) f^\nu\Big|(x) \ &= \ \Big|\Phi\left(D^{[p]}u^\nu(x)\right)\, F\Big(x, u^\nu(x), D^{[p]}u^\nu(x)\Big)\Big| \\
&= \int_{\overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s} \big|\Phi(\underline{\mathbf{X}})\, F\big(x, u^\nu(x), \underline{\mathbf{X}}\big)\big|\, d\big[\delta_{D^{[p]}u^\nu(x)}\big](\underline{\mathbf{X}}) \\
&\longrightarrow 0,
\end{aligned}$$

as $\nu \to \infty$, for a.e. $x \in \Omega$. The theorem follows. $\square$

Now is remains to establish Corollary 12.

Proof of Corollary 12. We continue from the proof of Theorem 11. First we note that the equivalences among the modes of convergence described in (3.32)-(3.33) are quite elementary. By choosing $\Phi \geq \chi_{\mathbb{B}_R(0)}$ or $\Phi \leq C\chi_{\mathbb{B}_R(0)}$ we see that (3.32) is equivalent to (3.33)(a). Then, (3.33)(a) is equivalent to (3.33)(b) by noting that a.e. convergence is equivalent (up to the passage to a subsequence) to convergence locally in measure and also by observing the identity

$$\Big\{|f^\nu|\, \chi_{\mathbb{B}_{1/\varepsilon}(0)}\big(D^{[p]}u^\nu\big) > \varepsilon\Big\} \ = \ \big\{|f^\nu| > \varepsilon\big\} \cap \Big\{\big|D^{[p]}u^\nu\big| < 1/\varepsilon\Big\}$$

which is valid for any $\varepsilon > 0$. Hence, in order to conclude it suffices to establish (3.34) when the set $E \subseteq \Omega$ is given by (3.35). To this end, we fix $\Omega' \Subset \Omega$ and we consider the sequence

$$\psi^\nu(x) \ := \ \min\big\{|f^\nu(x)|\, \chi_{\Omega' \setminus E}(x),\, 1\big\}, \qquad x \in \Omega,\ \nu \in \mathbb{N}.$$

Note now that $(\psi^\nu)_1^\infty$ is bounded and equi-integrable in $L^1(\Omega)$ because

$$\lim_{t \to \infty} \left(\sup_{\nu \in \mathbb{N}} \int_{\Omega \cap \{|\psi^\nu| > t\}} |\psi^\nu(x)|\, dx\right) \ = \ 0.$$

Moreover, the integral

$$\int_{\Omega' \setminus E} \int_{\mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nn^p}_s} \min\Big\{\big|F\big(x, u(x), \underline{\mathbf{X}}\big)\big|, 1\Big\}\, d\big[\mathcal{D}^{[p]}u(x)\big](\underline{\mathbf{X}})\, dx$$

exists because for a.e. $x \in \Omega \setminus E$, we have that $\big[\mathcal{D}^k u(x)\big](\{\infty\}) = 0$ for all $k = 1, ..., p$. In addition, by utilising that $u^\nu \longrightarrow u$ a.e. on $\Omega \setminus E$ and Lemma 16, we have

$$\delta_{(u^\nu, D^{[p]}u^\nu)} \overset{*}{\rightharpoondown} \delta_u \times \mathcal{D}^{[p]}u \quad \text{ in } \mathscr{Y}\Big(\Omega \setminus E,\, \mathbb{R}^N \times \mathbb{R}^{Nn} \times ... \times \mathbb{R}^{Nn^p}_s\Big),$$

as $\nu \to \infty$. Hence, by invoking Corollary 3.36 on p. 207 of [FG], we have

$$\lim_{\nu\to\infty}\int_{\Omega'\setminus E}\big|\psi^\nu(x)\big|\,dx$$
$$= \lim_{\nu\to\infty}\int_{\Omega'\setminus E}\min\Big\{\Big|F\Big(x,u^\nu(x),D^{[p]}u^\nu(x)\Big)\Big|,1\Big\}\,dx \tag{3.36}$$
$$= \int_{\Omega'\setminus E}\int_{\mathbb{R}^{Nn}\times\cdots\times\mathbb{R}^{Nn^p}_s}\min\Big\{\big|F\big(x,u(x),\underline{\mathbf{X}}\big)\big|,1\Big\}\,d\big[\mathcal{D}^{[p]}u(x)\big](\underline{\mathbf{X}})\,dx.$$

Since u is a $\mathcal{D}$-solution, we have

$$\sup_{\underline{\mathbf{X}}\in\,\mathrm{supp}_*(\mathcal{D}^{[p]}u(x))}\big|F\big(x,u(x),\underline{\mathbf{X}}\big)\big| \,=\, 0 \qquad \text{a.e. on } \Omega.$$

Moreover, for any $\varepsilon\in(0,1)$ we have the inequality

$$\varepsilon\Big|(\Omega'\setminus E)\cap\big\{|f^\nu|>\varepsilon\big\}\Big| \,\leq\, \int_{\Omega'\setminus E}\min\big\{|f^\nu(x)|,1\big\}\,dx$$
$$= \int_{\Omega}\big|\psi^\nu(x)\big|\,dx$$

Further, we note that for a.e. $x\in\Omega\setminus E$, $\mathcal{D}^{[p]}u(x)$ is a probability measure on $\mathbb{R}^{Nn}\times\cdots\times\mathbb{R}^{Nn^p}_s$ (not just on the compactification). By recalling the definition of the function ψ^ν, (3.36) and the above observations give

$$\varepsilon\limsup_{\nu\to\infty}\Big|(\Omega'\setminus E)\cap\big\{|f^\nu|>\varepsilon\big\}\Big|$$
$$\leq \int_{\Omega'\setminus E}\int_{\mathbb{R}^{Nn}\times\cdots\times\mathbb{R}^{Nn^p}_s}\min\Big\{\big|F\big(x,u(x),\underline{\mathbf{X}}\big)\big|,1\Big\}\,d\big[\mathcal{D}^{[p]}u(x)\big](\underline{\mathbf{X}})\,dx$$
$$\leq \int_{\Omega'\setminus E}\int_{\mathbb{R}^{Nn}\times\cdots\times\mathbb{R}^{Nn^p}_s}\big|F\big(x,u(x),\underline{\mathbf{X}}\big)\big|\,d\big[\mathcal{D}^{[p]}u(x)\big](\underline{\mathbf{X}})\,dx$$
$$\leq \int_{\Omega'\setminus E}\left(\sup_{\underline{\mathbf{X}}\in\,\mathrm{supp}\left(\mathcal{D}^{[p]}u(x)\right)\cap\left(\mathbb{R}^{Nn}\times\cdots\times\mathbb{R}^{Nn^p}_s\right)}\big|F\big(x,u(x),\underline{\mathbf{X}}\big)\big|\right)dx$$
$$= \int_{\Omega'\setminus E}\left(\sup_{\underline{\mathbf{X}}\in\,\mathrm{supp}_*(\mathcal{D}^{[p]}u(x))}\big|F\big(x,u(x),\underline{\mathbf{X}}\big)\big|\right)dx$$
$$= 0.$$

Conclusively, we have obtained that $f^\nu\longrightarrow 0$ locally in measure on $\Omega\setminus E$ and hence up to a subsequence $f^\nu\longrightarrow 0$ a.e. on $\Omega\setminus E$ as $\nu\to\infty$. The corollary has been established. $\square$

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